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On Kinematical Invariances of the Equations of Motion

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Abstract

Invariance properties of the equations of motion are considered from the differential geometric viewpoint, on making use of vector fields and differential forms. It will be shown that the kinematical symmetries generated from first integrals and the invariance such as the dilation can be treated on an equal footing.

1. *Introduction*

The usual approach to the symmetry problem of a dynamical system starts with first integrals of the system (Ikeda, 1970). Symmetry properties, however, can be regarded as invariance properties of the equations of motion. From this point of view, the homogeneity of the equations of motion is an invariance property. The isotropic harmonic oscillator is a simple example with such a property. As to the dilatation invariance readers can refer to many papers, e.g. Currie, 1966, for the particle dynamics and Flata *et al.*, 1970, for the field theory. In the present work invariances of the equations of motion will be dealt with in two ways.

Let M be an m-dimensional Riemannian manifold endowed with the positive-definite metric tensor (g_{ij}). The equations of motion are given by

$$\frac{D^2x^i}{dt^2} = f^i \qquad (i = 1, 2, \ldots, m) \tag{1.1}$$

where (x^i) denotes a local coordinate system of M and D/dt absolute derivation along a curve. (f^i) is a given vector field of force. Equation (1.1) can be described on the tangent bundle TM or on the cotangent bundle T^*M. The methods of description may be called *Lagrangean* and *Hamiltonian formalisms* respectively, even if there is no Lagrangean or Hamiltonian function.

In Section 2 an approach to the invariance of a differential equation will be prepared for later use. In Sections 3 and 4 the *kinematical invariance* of the equations of motion will be discussed in Lagrangean and Hamiltonian formalisms respectively. The term '*kinematical*' is to imply being raised from transformations of M. Section 5 is concerned with the symmetry problem in the variational theory, e.g., Noether's theorem.

It will be found that the invariance property in Hamiltonian formalism is covered by that in Lagrangean formalism.

2. *An Invariance Algebra*

Let N be an n-dimensional differentiable* manifold and X a vector field defined on N. An integral curve of X is given by a solution curve of the differential equation (referred to the characteristic equation of X)

$$\frac{dx^\lambda}{dt} = X^\lambda \qquad (\lambda = 1, 2, \ldots, n) \tag{2.1}$$

where (x^λ) denotes a local coordinate system of N and (X^λ) the components of X with respect to (x^λ). Let Y be a vector field on N. If there is a function ρ on N such that

$$[Y, X] = \rho X \tag{2.2}$$

it is said that equation (2.1) admits an infinitesimal transformation Y (Cartan, 1932). So to speak, relation (2.2) shows the invariance property of equation (2.1). The set of all Y's $\neq X$ which satisfy (2.2) forms a Lie algebra, which may be called an *invariance algebra* of (2.1).

Since we are interested in the equations of motion, parameter t has to be regarded as time in the sense of Newtonian mechanics. Therefore we deal with

$$[Y, X] = 0 \tag{2.3}$$

excluding the case of $\rho \neq 0$ where transformations of t are involved. Affine transformations of t, $t' = at + b$, are still admitted by (2.3), but are not contradictory to the classical notion of time. An infinitesimal transformation Y satisfying (2.3) is called a *variational vector field* of X or (2.1).

3. *Lagrangean Formalism*

Let (x^i, v^i) be an induced coordinate system of the tangent bundle TM of a Riemannian manifold M. The equations of motion (1.1) can be translated into

$$\begin{cases} \dfrac{dx^i}{dt} = v^i \\ \\ \dfrac{dv^i}{dt} = -\left\{ {i \atop jk} \right\} v^j v^k + f^i \dagger \end{cases} \tag{3.1}$$

* Hereafter we will tacitly assume that all manifolds, vector fields, functions, etc., introduced have a suitable order of differentiability.

† Unless otherwise stated, Latin indices run from 1 to m and the summation convention is adopted.

where $\{^i_{jk}\}$ are the Christoffel symbols. Equation (3.1) is the characteristic equation of the vector field

$$Z = v^i \frac{\partial}{\partial x^i} + (f^i - \{^i_{jk}\} v^j v^k) \frac{\partial}{\partial v^i} \tag{3.2}$$

For a strict discussion, we have to verify that Z is defined independently of the choice of the local coordinate system. The verification, however, can be easily carried out. It is convenient to write the vector field (3.2) in the form

$$Z = E + f^i \frac{\partial}{\partial v^i} \tag{3.3}$$

where E is defined by

$$E = v^i \left(\frac{\partial}{\partial x^i} - \{^j_{ik}\} v^k \frac{\partial}{\partial v^j} \right) \tag{3.4}$$

and is called the *GF-vector field* (Sasaki, 1958).

Given a vector field on M

$$X = \xi^i \frac{\partial}{\partial x^i} \tag{3.5}$$

then its *lift* to *TM* (Sasaki, 1958) is defined by

$$\tilde{X} = \xi^i \frac{\partial}{\partial x^i} + \frac{\partial \xi^i}{\partial x^j} v^j \frac{\partial}{\partial v^i} \tag{3.6}$$

We are now in a position to study the condition that the equations of motion (1.1) admit an infinitesimal transformation (3.5) of M. To this end, we have only to verify

$$[\tilde{X}, Z] = 0 \tag{3.7}$$

where Z is given by (3.2) and $\tilde{X}$ by (3.6). The result is written as follows.

$$[\tilde{X}, Z] = (\mathscr{L}_X f^k) \frac{\partial}{\partial v^k} - (v^i v^j \mathscr{L}_X \{^k_{i\,j}\}) \frac{\partial}{\partial v^k} = 0 \tag{3.8}$$

where $\mathscr{L}_X$ denotes Lie derivation with respect to X. Thus we obtain the following theorem.

Theorem 1. *A necessary and sufficient condition for the equations of motion* (1.1) *to admit an infinitesimal transformation* (3.5) *is given by*

$$\mathscr{L}_X f^k = 0, \qquad \mathscr{L}_X \{^i_{jk}\} = 0 \tag{3.9}$$

That is to say, X is an infinitesimal affine transformation that leaves the force field (f^k) *invariant.*

The dilatation invariance of the isotropic harmonic oscillator is covered by this theorem. As is easily seen, if the dynamical system (TM, Z) is conservative, that is, $-f^k = g^{kj}(\partial U/\partial x^j)$, (g^{kj}) denoting the inverse of (g_{ij}), then (3.9) includes the condition for the kinematical symmetry (Ikeda, 1970):

$$\mathscr{L}_X U = 0, \qquad \mathscr{L}_X g_{ij} = 0$$

4. *Hamiltonian Formalism*

In classical Hamiltonian mechanics dynamical states are described by (x^i, p_i), where p_i is defined by $p_i = g_{ij}v^j$, and is called generalised momentum. So it can be said that Hamiltonian mechanics is described on the cotangent bundle T^*M. To make sure, we note that the cotangent bundle of a Riemannian manifold is diffeomorphic to the tangent bundle. And the diffeomorphism, φ: $TM \to T^*M$, is given by

$$x^i = x^i, \qquad p_i = g_{ij}v^j \tag{4.1}$$

in local coordinate systems. The dynamical law given by (3.2) is transformed onto the cotangent bundle by the differential φ_* of φ.

Lemma 1. *Given Z defined by* (3.2). *Then the transformed vector field φ_*Z on T^*M is given by*

$$\varphi_*Z = \frac{\partial T}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial T}{\partial x^i}\frac{\partial}{\partial p_i} + f_i\frac{\partial}{\partial p_i} \tag{4.2}$$

where $f_i = g_{ij}f^j$ and

$$T = \tfrac{1}{2}g^{ij}p_ip_j \tag{4.3}$$

Proof. Let E be the GF-vector field defined by (3.4). Direct calculation gives

$$\varphi_*E = \frac{\partial T}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial T}{\partial x^i}\frac{\partial}{\partial p_i} \tag{4.4}$$

$$\varphi_*\left(f^i\frac{\partial}{\partial v^i}\right) = f_i\frac{\partial}{\partial p_i}$$

Hereafter we put $Z^* = \varphi_*Z$.

Lemma 1 shows that the well-known fact that the time development of the Hamiltonian system is subject to the equations

$$\begin{cases} \dfrac{dx^i}{dt} = \dfrac{\partial T}{\partial p_i} \\[2ex] \dfrac{dp_i}{dt} = -\dfrac{\partial T}{\partial x^i} + f_i \end{cases} \tag{4.5}$$

The *lift* of the vector field X (3.5) on M to the cotangent bundle T^*M (Yano, 1967) is defined by

$$X^* = \xi^i \frac{\partial}{\partial x^i} - \frac{\partial \xi^k}{\partial x^i} p_k \frac{\partial}{\partial p_i} \tag{4.6}$$

Here we give a lemma.

Lemma 2. *Let $\tilde{X}$ and X^* be the lifts defined by* (3.6) *and* (4.6) *respectively. Let φ denote the diffeomorphism given by* (4.1). *A necessary and sufficient condition for $\varphi_* \tilde{X} = X^*$ is that X is a Killing vector field.*

Proof. Simple calculation gives

$$\varphi_* \tilde{X} = \xi^i \frac{\partial}{\partial x^i} + g^{kl} p_l \left(\xi^j \frac{\partial g_{ik}}{\partial x^j} + g_{ij} \frac{\partial \xi^j}{\partial x^k} \right) \frac{\partial}{\partial p_i} \tag{4.7}$$

Assertion in the lemma is clear from (4.6) and (4.7).

If $\tilde{X}$ is a variational vector field of Z, $[\tilde{X}, Z] = 0$ holds good. Since φ is a diffeomorphism, we have $[\varphi_* \tilde{X}, \varphi_* Z] = 0$. Therefore, from Lemma 2, the sufficient condition of the following theorem is satisfied.

Theorem 2. *X^* defined by* (4.6) *is a variational vector field of Z^* defined by* (4.2), *if and only if*

$$\mathscr{L}_X f_i = 0, \qquad \mathscr{L}_X g_{ij} = 0 \tag{4.8}$$

Proof. If we calculate (2.3) for X^* and Z^*, we obtain

$$\begin{aligned} [X^*, Z^*] = & -\frac{\partial}{\partial p_j} ((\nabla_i \xi^k) g^{il} p_k p_l + f_i \xi^i) \frac{\partial}{\partial x^j} \\ & + \left(\frac{\partial}{\partial x^j} ((\nabla_i \xi^k) g^{il} p_k p_l) + \xi^l \frac{\partial f_j}{\partial x^l} + f_l \frac{\partial \xi^l}{\partial p_j} \right) \frac{\partial}{\partial p_j} = 0 \end{aligned} \tag{4.9}$$

From the first terms of (4.9) it follows that

$$\nabla_i \xi_k + \nabla_k \xi_i = 0$$

Substituting this into the last terms, we have

$$\mathscr{L}_X f_j = \xi^l \frac{\partial f_j}{\partial x^l} + f_l \frac{\partial \xi^l}{\partial x^j} = 0$$

As is mentioned in Section 1, Theorem 2 is covered by Theorem 1. We note that Lemma 2 explains the difference between Theorem 1 and Theorem 2.

The lift X^* defined by (4.6) can be regarded as the infinitesimal canonical transformation

$$X_F = \frac{\partial F}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial F}{\partial x^i} \frac{\partial}{\partial p_i}$$

generated from $F = \xi^i p_i$. For a general infinitesimal canonical transformation we have the following theorem.

Theorem 3. *The infinitesimal canonical transformation generated from a function S on T^*M is the variational vector field of Z^*, if and only if the following equations are satisfied by S.*

$$\begin{cases} \dfrac{\partial}{\partial p_k}\left(\{S, T\} + f_i \dfrac{\partial S}{\partial p_i}\right) = 0 \\ \dfrac{\partial}{\partial x^k}\left(\{S, T\} + f_i \dfrac{\partial S}{\partial p_i}\right) = \dfrac{\partial S}{\partial p_j}\left(\dfrac{\partial f_j}{\partial x^k} - \dfrac{\partial f_k}{\partial x^j}\right) \end{cases} \tag{4.10}$$

where T is the function defined by (4.3) *and* $\{S, T\}$ *denotes the Poisson bracket between S and T.*

Proof. We have obtained $\varphi_* E = X_T$ in (4.4), where X_T denotes the infinitesimal canonical transformation generated from T. We note the formula

$$[X_S, X_T] = -X_{\{S,\, T\}} \tag{4.11}$$

From these facts we obtain

$$\begin{aligned} [X_S, Z^*] &= -\frac{\partial}{\partial p_k}\left(\{S, T\} + f_i \frac{\partial S}{\partial p_i}\right)\frac{\partial}{\partial x^k} \\ &+ \left(\frac{\partial}{\partial x^k}\left(\{S, T\} + f_i \frac{\partial S}{\partial p_i}\right) + \frac{\partial S}{\partial p_j}\left(\frac{\partial f_k}{\partial x^j} - \frac{\partial f_j}{\partial x^k}\right)\right)\frac{\partial}{\partial p_k} \end{aligned} \tag{4.12}$$

This completes the proof.

Theorem 4. *If the dynamical system* (T^*M, Z^*) *is conservative, that is* $f_i = -(\partial U/\partial x^i)$, *then* (4.10) *is reduced to*

$$\{S, H\} = \text{const.} \tag{4.13}$$

where H is the Hamiltonian defined by

$$H = T + U \tag{4.14}$$

Proof. Since the system is conservative, Z^* is rewritten in the form

$$Z^* = X_T - \frac{\partial U}{\partial x^i}\frac{\partial}{\partial p_i} = X_H$$

Consequently (4.12) is reduced to

$$[X_S, Z^*] = [X_S, X_H] = -X_{\{S,\, H\}} = 0$$

and (4.13) is obtained.

5. *Relation to the Variational Theory*

In what follows we deal with the invariance property developed in Section 3 in terms of differential forms. Such a way of discussion is a variational approach to the invariance.

All instruments are set on the product $TM \times \mathbf{R}$. (x^i, v^i, t) is adopted as a local coordinate system. The 2-form defined by

$$\omega = d(g_{ij}v^j dx^i - T dt) + f_i dx^i \wedge dt \tag{5.1}$$

is called the generating form of the equations of motion, where

$$T = \tfrac{1}{2} g_{ij} v^i v^j \tag{5.2}$$

If we introduce a notation Dv^i, covariant derivative of v^i, by

$$Dv^i = dv^i + \left\{ {i \atop jk} \right\} v^k dx^j \tag{5.3}$$

we can rewrite (5.1) in the form

$$\omega = g_{ij}(Dv^j - f^j dt) \wedge (dx^i - v^i dt) \tag{5.4}$$

It is clear that the definition of ω is independent of the choice of local coordinate system.

If a vector field Y on $TM \times \mathbf{R}$ satisfies

$$i(Y)\omega = 0, \qquad Yt = 1 \tag{5.5}$$

where $i(Y)$ denotes interior product by Y, Y is called the characteristic vector field of ω. The characteristic equation of Y is called the characteristic equation of ω. The reason for the name of generating form is given by the following theorem.

Theorem 5 (*Hamilton's principle*). *The equation of motion is given by the characteristic equation of the generating form ω.*

Proof. As is easily seen,

$$Z = v^i \left(\frac{\partial}{\partial x^i} - \left\{ {j \atop i\,k} \right\} v^k \frac{\partial}{\partial v^j} \right) + f^i \frac{\partial}{\partial v^i} + \frac{\partial}{\partial t} \tag{5.6}$$

is the only solution to (5.5) and coincides with (3.2) up to $\partial/\partial t$.

The following lemma is trivial.

Lemma 3. *A necessary and sufficient condition for the generating form ω to be closed is that $f_i dx^i$ is closed, so that the system is conservative.*

A *first integral* is defined as a 1-form π on $TM \times \mathbf{R}$ satisfying

$$d\pi = 0, \qquad \pi(Z) = 0 \tag{5.7}$$

where Z is the vector field given by (5.6). The following theorem is a differential geometric version of the so-called Noether's theorem.

Theorem 6 (*Gallissot*, 1954). *Let ω be closed. If Y is an infinitesimal automorphism of ω, that is, $\mathcal{L}_Y \omega = 0$, $i(Y)\omega$ is a first integral. Conversely if π is a first integral, there is a vector field Y such that $i(Y)\omega = \pi$, and Y is an infinitesimal automorphism of ω. (Y is called to be generated from π.)*

Let $\tilde{X}$ and ω be the vector field (3.6) and the generating form (5.4) respectively. We calculate the Lie derivative of ω with respect to $\tilde{X}$ we obtain

$$\begin{aligned}\mathscr{L}_{\tilde{X}}\omega = (\mathscr{L}_X g_{ij})\,(Dv^j - f^j dt) \wedge (dx^i - v^i dt) \\ + g_{ij}(\mathscr{L}_X\{{}^{\,j}_{k\,l}\}\, v^l dx^k - \mathscr{L}_X f^j dt) \wedge (dx^i - v^i dt)\end{aligned} \tag{5.8}$$

This result is very useful for our discussion. We can attain Theorem 1 by virtue of (5.8). In fact, if $\tilde{X}$ leaves the equations of motion invariant, Z must be a characteristic vector field of $\mathscr{L}_{\tilde{X}}\,\omega$. And we have

$$\mathscr{L}_X\{{}^{\,j}_{k\,l}\}\, v^l v^k - \mathscr{L}_X f^j = 0$$

Thus we are led to (3.9). Conversely if (3.9) holds, Z is a characteristic vector field of (5.8) together with (3.9).

We assume that ω is closed, so that the system is conservative. Then we can say that the invariance discussed in Section 3 is not necessarily generated from first integrals. In fact, if $\tilde{X}$ is generated from a first integral, it must be an infinitesimal automorphism of ω (Theorem 6), and X is a Killing vector field by virtue of (5.8). Therefore it follows that if X is not a Killing vector field $\tilde{X}$ cannot be generated from a first integral. We have known an infinitesimal homothetic transformation which is not a Killing vector field (the dilatation invariance).

Retaining the assumption for ω, we use (5.8) to show one theorem. As a definition, Y is an infinitesimal conformal transformation of ω if there is a function ρ on $TM \times \mathbf{R}$ such that $\mathscr{L}_Y\omega = \rho\omega$.

Theorem 7. *Assume that M is connected, $m > 1$ and ω is closed. $\tilde{X}$ given by* (3.6) *is an infinitesimal conformal transformation of ω given by* (5.4), *if and only if the following conditions are satisfied.*

$$\mathscr{L}_X g_{ij} = \rho\, g_{ij}, \qquad \mathscr{L}_X f^i = 0, \qquad \rho = \text{const.} \tag{5.9}$$

That is to say, X is an infinitesimal homothetic transformation that leaves the field (f^i) invariant.

Proof. Suppose that $\tilde{X}$ is an infinitesimal conformal transformation. Under the assumption of the theorem, it is known that ρ is constant (Takizawa, 1963). On the other hand (5.8) gives

$$\mathscr{L}_X g_{ij} = \rho g_{ij}, \qquad \mathscr{L}_X\{{}_j{}^{\,i}{}_k\} = \mathscr{L}_X f^l = 0$$

Since a homothetic transformation is affine, (5.9) is obtained. Converse is clear from (5.8).

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